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Separation of variables in the Kramers equation

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Abstract. We consider the problem of separation of variables in the Kramers equation admitting a non-trivial symmetry group. Provided the external potential $V(x)$ is at most quadratic, a complete solution of the problem of separation of variables is obtained. Furthermore, we construct solutions of the Kramers equation with separated variables in explicit form.

1. Introduction

Many phenomena in physics and, especially, in chemical physics may be modelled as the Brownian motion of particles in an external potential $V(x)$, the appropriate transport equation being the (1 + 2)-dimensional Fokker–Planck equation of special form

$$u_t = \nu u_{yy} - y u_x + (\nu y + V'(x)) u_y + \nu u \tag{1}$$

where $u = u(t, x, y)$ is a sufficiently smooth real valued function and ν is a real parameter.

The first relevant result on studying the partial differential equation (PDE) (1) has been obtained by Kramers [1]. He found a solution of the escape problem of a classical particle subjected to Gaussian white noise out of a deep potential well. This is why the equation in question is called the Kramers equation (KE) (see, for more details [2–4]).

As KE is a PDE with variable coefficients, we cannot apply the Fourier transform in order to solve it. In fact the only way to obtain exact solutions of KE are either to utilize its Lie symmetry or to apply the method of the separation of variables. The first possibility has been exploited recently in [5, 6], where symmetry classification of the class of PDEs (1) has been carried out. The principal result of these papers is that KE has a symmetry group that is wider than a trivial one-parameter group of time translations if and only if $V''(x) = 0$.

The principal aim of this paper is to apply the direct approach to variable separation in PDEs suggested in [7–9] to solve KE. As is well known, separability of the PDE is intimately connected to its symmetry within the class of second-order differential operators [10]. This is why, we will concentrate on the case $V(x) = kx$, $k =$ constant, namely, we consider the KE having non-trivial Lie symmetry

$$u_t = \nu u_{yy} - y u_x + (\nu y + kx) u_y + \nu u. \tag{2}$$

In a classical setting the method of separation of variables (say, in the Cartesian coordinate system) is based on a special representation of a solution to be found in factorized form

$$u(t, x, y) = \varphi_0(t)\varphi_1(x)\varphi_2(y)$$

where φ_i, $i = 0, 1, 2$, are solutions of some ordinary differential equations (ODEs). However, one can try to separate variables in this equation in another coordinate system, for example, in polar coordinates and look for a solution of the form

$$u(t, x, y) = \varphi_0(t)\varphi_1(\sqrt{x^2 + y^2})\varphi_2\left(\arctan \frac{y}{x}\right).$$

So, if we are given any coordinate system, then it is clear how to get exact solutions with separated variables. However, the classical approach gives no general routine for finding all possible coordinate systems providing separability of the equation. Our approach to the problem of the separation of variables in evolution-type equations (to be specific, we take the case of an equation having three independent variables t, x, y) is based on the following observations.

- All solutions with separated variables known to us can be represented in the form

$$u(t, x, y) = Q(t, x, y)\varphi_0(t)\varphi_1(\omega_1(t, x, y))\varphi_2(\omega_2(t, x, y)) \tag{3}$$

  where Q, ω_1, ω_2 are sufficiently smooth functions and φ_i, $i = 0, 1, 2$ satisfy some ODEs.
- The functions φ_i, $i = 0, 1, 2$, depend on two arbitrary parameters λ_1, λ_2 called spectral parameters or separation constants. Furthermore, the functions Q, ω_1, ω_2 are independent of λ_1, λ_2.

By properly postulating these features we have formulated an efficient approach to the problem of variable separation in linear PDEs [9]. Applying it to the KE (2) we look for its particular solutions of the form (3), where functions Q, ω_1, ω_2 are chosen in such a way that inserting (3) into KE yields three ODEs for functions $\varphi_0(t), \varphi_1(\omega_1), \varphi_2(\omega_2)$

$$\begin{aligned} &U_0(t, \varphi_0, \dot{\varphi}_0; \lambda_1, \lambda_2) = 0 \\ &U_i(\omega_i, \varphi_i, \dot{\varphi}_i, \ddot{\varphi}_i; \lambda_1, \lambda_2) = 0 \qquad i = 1, 2. \end{aligned} \tag{4}$$

Here U_0, U_1, U_2 are some smooth functions of the indicated variables, λ_1, λ_2 are real parameters and

$$\operatorname{rank} \left\| \begin{matrix} \dfrac{\partial U_0}{\partial \lambda_1} & \dfrac{\partial U_0}{\partial \lambda_2} \\ \dfrac{\partial U_1}{\partial \lambda_1} & \dfrac{\partial U_1}{\partial \lambda_2} \\ \dfrac{\partial U_2}{\partial \lambda_1} & \dfrac{\partial U_2}{\partial \lambda_2} \end{matrix} \right\| = 2. \tag{5}$$

Note that the functions Q, ω_1, ω_2 are independent of λ_1, λ_2.

Provided these requirements are met, we say that KE is separable in the coordinate system $t, \omega_1(t, x, y), \omega_2(t, x, y)$.

Due to the fact that the equation under study is linear, the reduced equations prove to be linear as well. Furthermore, we have to consider two distinct cases.

Case 1. The system of equations (4) has the form

$$\begin{aligned} \dot{\varphi}_0 &= A_0(t; \lambda_1, \lambda_2)\varphi_0 \\ \dot{\varphi}_1 &= A_1(\omega_1; \lambda_1, \lambda_2)\varphi_1 \\ \ddot{\varphi}_2 &= A_2(\omega_2; \lambda_1, \lambda_2)\dot{\varphi}_2 + A_3(\omega_2; \lambda_1, \lambda_2)\varphi_2. \end{aligned} \tag{6}$$

Case 2. The system of equations (4) has the form

$$\begin{aligned}\dot\varphi_0 &= A_0(t;\lambda_1,\lambda_2)\varphi_0\\ \dot\varphi_1 &= A_1(\omega_1;\lambda_1,\lambda_2)\varphi_1\\ \dot\varphi_2 &= A_2(\omega_2;\lambda_1,\lambda_2)\varphi_2.\end{aligned} \tag{7}$$

In these formulae $A_0, \ldots, A_3$ are some smooth real valued functions of the indicated variables.

Consequently, there are two different means to separate variables in KE, either to reduce it to two first-order and one second-order ODEs or to three first-order ODEs. It is impossible to reduce KE to two or three second-order ODEs because it contains a second-order derivative with respect to one variable only.

Provided the system of reduced ODEs has the form (6), separation of variables in (2) is performed in the following way.

1. We insert the ansatz (3) into KE and express the derivatives $\dot\varphi_0$, $\dot\varphi_1$, $\ddot\varphi_1$, $\ddot\varphi_2$ in terms of functions $\varphi_0, \varphi_1, \varphi_2, \dot\varphi_2$ using equations (6) and their differential consequences (where necessary).
2. The equality obtained is split by $\varphi_0, \varphi_1, \varphi_2, \dot\varphi_2, \lambda_1, \lambda_2$ which are regarded as independent variables. This yields an over-determined system of nonlinear PDEs for unknown functions Q, ω_1, ω_2.
3. After solving this system we get an exhaustive description of coordinate systems providing separability of KE.

Clearly, if we adopt a more general definition of the separation of variables, then additional coordinate systems providing separability of KE may appear. However, all solutions with separated variables of the Schrödinger and heat conductivity equations known to us can be obtained within the described approach.

The case when the system of reduced ODEs is of the form (7) is handled in a similar way.

Next, we introduce an equivalence relation on the set of all coordinate systems providing separability of KE. We say that two coordinate systems t, ω_1, ω_2 and t', ω_1', ω_2' are equivalent if the corresponding solutions with separated variables are transformed one into another by

- the group transformations from the Lie transformation group admitted by KE,
- the transformations of the form

$$t \to t' = f_0(t) \qquad \omega_i \to \omega_i' = f_i(\omega_i) \tag{8}$$

$$Q \to Q' = Qh_0(t)h_1(\omega_1)h_2(\omega_2) \tag{9}$$

where f_0, f_i, h_0, h_i are some smooth functions.

It can be proved that formulae (8), (9) define the most general transformation preserving the class of ansätze (3). The equivalence relation splits the set of all possible coordinate systems into equivalence classes. In a sequel, when presenting the lists of coordinate systems enabling us to separate variables in KE we will give only one representative for each equivalence class.

2. Principal results

In this section we give a complete account of our results on the separation of variables in KE obtained within the framework of the approach described in the introduction. We write down explicit forms of the functions $Q(t,x,y)$, $\omega_1(t,x,y)$, $\omega_2(t,x,y)$ and the corresponding reduced ODEs for functions $\varphi_0(t)$, $\varphi_1(\omega_1)$, $\varphi_2(\omega_2)$.

Theorem 1. *Equation (2) admits the separation of variables into two first-order and one second-order ODEs if and only if* k *takes one of the three values* 0, $3\nu^2/16$, $-3\nu^2/4$. *Furthermore, equations separate into three first-order ODEs with arbitrary* k.

Theorem 1 gives a general description of separable KEs. The solution of the problem of separation of variables in corresponding KEs is provided by theorems 2–6 later.

Theorem 2. *The set of inequivalent coordinate systems providing separability of KE with* $k = \nu^2/4$ *is exhausted by the following ones*

$$\omega_i = \frac{f_i y - \dot{f}_i x}{\dot{f}_2 f_1 - \dot{f}_1 f_2} \qquad i = 1, 2$$

$$Q = \exp\left\{\left(-\frac{1}{4\nu}\frac{\ddot{f}_2 f_1 - \ddot{f}_1 f_2}{\dot{f}_2 f_1 - \dot{f}_1 f_2} - \frac{1}{4}\right)y^2 + \frac{1}{2\nu}\left(\frac{\ddot{f}_2 \dot{f}_1 - \ddot{f}_1 \dot{f}_2}{\dot{f}_2 f_1 - \dot{f}_1 f_2} - k\right)x \right.$$
$$\left. + \left(\frac{1}{4\nu}\frac{\dddot{f}_2\,\dot{f}_1 - \dddot{f}_1\,\dot{f}_2}{\dot{f}_2 f_1 - \dot{f}_1 f_2} - \frac{k}{4}\right)x^2 - \frac{1}{2}\ln|\dot{f}_2 f_1 - \dot{f}_1 f_2| + \frac{\nu}{2}t\right\}$$

$$\dot{\varphi}_0 = \nu\left(\frac{f_1\lambda_1 + f_2\lambda_2}{\dot{f}_2 f_1 - \dot{f}_1 f_2}\right)^2 \varphi_0 \qquad \dot{\varphi}_1 = \lambda_1\varphi_1 \qquad \dot{\varphi}_2 = \lambda_2\varphi_2 \tag{10}$$

where

$$f_1 = t\left(A_1 \sinh\frac{\nu}{2}t + A_2\cosh\frac{\nu}{2}t\right) + A_3\sinh\frac{\nu}{2}t + A_4\cosh\frac{\nu}{2}t$$
$$f_2 = t\left(B_1 \sinh\frac{\nu}{2}t + B_2\cosh\frac{\nu}{2}t\right) + B_3\sinh\frac{\nu}{2}t + B_4\cosh\frac{\nu}{2}t$$

and $A_1, \ldots, B_4$ *are arbitrary real constants satisfying the condition* $2C_{12} - \nu(C_{13} - C_{24}) = 0$. *Hereafter we use the notations*

$$C_{ij} = B_i A_j - A_i B_j \qquad i, j = 1, \ldots, 4.$$

Theorem 3. *The set of inequivalent coordinate systems providing separability of KE with* $k > \nu^2/4$ *is exhausted by those given in (10) with*

$$f_1 = \sin bt(A_1 \sinh at + A_2\cosh at) + \cos bt(A_3\sinh at + A_4\cosh at)$$
$$f_2 = \sin bt(B_1 \sinh at + B_2\cosh at) + \cos bt(B_3\sinh at + B_4\cosh at)$$

where $a = \nu/2, b = (k - (\nu^2/4))^{1/2}$ *and* $A_1, \ldots, B_4$ *are constants fulfilling the condition* $(C_{12} + C_{34})b + (C_{13} - C_{24})a = 0$. *The explicit form of the function* Q *and the reduced ODEs are also obtained from the formulae (10) with* f_1, f_2 *given previously.*

Theorem 4. *The set of inequivalent coordinate systems providing separability of KE with* $k < \nu^2/4$ *and* $k \neq 0, 3\nu^2/16, -3\nu^2/4$ *is exhausted by those given in (10) with*

$$f_1 = \sinh bt(A_1 \sinh at + A_2\cosh at) + \cosh bt(A_3\sinh at + A_4\cosh at)$$
$$f_2 = \sinh bt(B_1 \sinh at + B_2\cosh at) + \cosh bt(B_3\sinh at + B_4\cosh at)$$

where $a = \nu/2, b = ((\nu^2/4) - k)^{1/2}$ *and* $A_1, \ldots, B_4$ *are constants fulfilling the condition* $(C_{12} - C_{34})b + (C_{13} - C_{24})a = 0$. *The explicit form of the function* Q *and reduced ODEs are also obtained from the formulae (10) with* f_1, f_2 *given before.*

Theorem 5. *The set of inequivalent coordinate systems providing separability of KE with $k = 0$ is exhausted by:*

(1) *those given in (10) with*

$$f_1 = A_1 \sinh \nu t + A_2 \cosh \nu t + A_3 t + A_4$$
$$f_2 = B_1 \sinh \nu t + B_2 \cosh \nu t + B_3 t + B_4$$

where $A_1, \ldots, B_4$ are constants fulfilling the equation $\nu C_{12} - C_{34} = 0$. The explicit form of the function Q and reduced ODEs are also obtained from the formulae (10) with f_1, f_2 given previously;

(2) *the following coordinate system*

$$\omega_1 = x \qquad \omega_2 = y \qquad Q = \exp\left(-\frac{y^2}{4}\right)$$
$$\dot\varphi_0 = \nu\lambda_1\varphi_0 \qquad \dot\varphi_1 = \nu\lambda_2\varphi_1 \qquad \ddot\varphi_2 = \left(\frac{y^2}{4} + \lambda_2 y + \lambda_1 - \frac{1}{2}\right)\varphi_2.$$

Theorem 6. *The set of inequivalent coordinate systems providing separability of KE with $k = 3\nu^2/16$ or $k = -3\nu^2/4$ is exhausted by*

(1) *those given in theorem 4 under $k = 3\nu^2/16$ or $k = -3\nu^2/4$;*

(2) *the following coordinate systems*

$$\omega_1 = R^3 x \qquad \omega_2 = Ry + 3\dot R x$$
$$Q = \exp\left\{\left(\frac{\dot R}{\nu R} - \frac{1}{4}\right)y^2 + \frac{1}{2\nu}\left(3\frac{\ddot R}{R} - k\right)xy + \left(-\frac{3}{4\nu}\frac{\dddot R}{R} + \frac{15\dot R\ddot R}{4\nu R^2} - \frac{k}{4}\right)x^2 + \frac{\nu}{2}t + 2\ln R\right\}$$
$$\dot\varphi_0 = \nu\lambda_1 R^2\varphi_0 \qquad \dot\varphi_1 = \nu\lambda_2\varphi_1 \qquad \ddot\varphi_2 = (\lambda_2\omega_2 + \lambda_1)\varphi_2$$

where

$$R(t) = \begin{cases} \dfrac{1}{\cosh at} \\ \dfrac{1}{\sinh at} \\ \exp\{\pm at\} \end{cases} \qquad \text{with } a = \begin{cases} \dfrac{\nu}{4} & \text{under } k = \dfrac{3\nu^2}{16} \\ \dfrac{\nu}{2} & \text{under } k = -\dfrac{3\nu^2}{4}. \end{cases}$$

3. Proof of theorems 1–6

In order to prove the assertions of the previous section one should apply to equation (2) the algorithm of variable separation described in the introduction.

We give a detailed proof for the case when the system of reduced ODEs is of the form (6). Inserting ansatz (3) into KE (2) and expressing the derivatives $\dot\varphi_0$, $\dot\varphi_1$, $\ddot\varphi_1$, $\ddot\varphi_2$ in terms of the functions $\varphi_0, \varphi_1, \varphi_2, \dot\varphi_2$ with the use of equations (6) and their differential consequences yield a system of two nonlinear PDEs

$$Q\omega_{2t} + yQ\omega_{2x} = \nu(yQ\omega_{2y} + 2Q_y\omega_{2y} + 2QA_1\omega_{1y}\omega_{2y} + QA_2\omega_{2y}^2 + Q\omega_{2yy}) + kxQ\omega_{2y} \tag{11}$$

$$\begin{aligned} Q_t + QA_0 + QA_1\omega_{1t} + yQ_x + yQA_1\omega_{1x} &= \nu(Q + yQ_y + yQA_1\omega_{1y} + Q_{yy} + 2Q_yA_1\omega_{1y} + Q(A_1^2 + A_{1\omega_1})\omega_{1y}^2 \\ &\quad + QA_1\omega_{1yy} + QA_3\omega_{2y}^2) + kx(Q_y + QA_1\omega_{1y}). \end{aligned} \tag{12}$$

This system is split with respect to variables λ_1, λ_2 (we recall that the functions ω_1, ω_2 are independent of λ_1, λ_2). To this end we differentiate (11) with respect to λ_i and get

$$(2A_{1\lambda_i}\omega_{1y} + A_{2\lambda_i}\omega_{2y})\omega_{2y} = 0 \qquad i = 1, 2.$$

Due to the fact that ω_{2y} does not vanish identically (otherwise it follows from (11) that $\omega_2 =$ constant), the equation

$$2A_{1\lambda_i}\omega_{1y} + A_{2\lambda_i}\omega_{2y} = 0 \qquad i = 1, 2 \tag{13}$$

holds.

Let us show first that we can, without loss of generality, put $\omega_{1y} = 0$. Suppose the inverse, namely that the inequality $\omega_{1y} \neq 0$, holds true. It follows from the second equation of system (6) that $A^2_{1\lambda_1} + A^2_{1\lambda_2} \neq 0$. Let the function $A_{1\lambda_1}$ be non-vanishing, then by the influence of (13) $A_{2\lambda_1} \neq 0$. Denoting

$$A_{1\lambda_1} = g(\omega_1, \lambda_1, \lambda_2) \qquad -2A_{2\lambda_1} = f(\omega_2, \lambda_1, \lambda_2)$$

we rewrite (13) as follows

$$\frac{\omega_{1y}}{\omega_{2y}} = \frac{f(\omega_2, \lambda_1, \lambda_2)}{g(\omega_1, \lambda_1, \lambda_2)}. \tag{14}$$

Differentiating (14) with respect to λ_1 yields

$$\frac{f_{\lambda_1}}{f} = \frac{g_{\lambda_1}}{g}.$$

Hence, we conclude that there is a function $k = k(\lambda_1, \lambda_2)$ such that

$$\frac{f_{\lambda_1}}{f} = \frac{g_{\lambda_1}}{g} = k(\lambda_1, \lambda_2).$$

Integrating these equations we get

$$f = k_1(\lambda_1, \lambda_2) f_1(\omega_2, \lambda_2) \qquad g = k_1(\lambda_1, \lambda_2) g_1(\omega_1, \lambda_1)$$

so that (14) reduces to the relation

$$\frac{\omega_{1y}}{\omega_{2y}} = \frac{f_1(\omega_2, \lambda_2)}{g_1(\omega_1, \lambda_2)}.$$

In a similar way we establish that the last relation is equivalent to the following one

$$\frac{\omega_{1y}}{\omega_{2y}} = \frac{f_2(\omega_2)}{g_2(\omega_1)}$$

hence

$$g_2(\omega_1)\omega_{1y} = f_2(\omega_2)\omega_{2y}.$$

Taking into account the equivalence relation (8) we can put $g_2 = 1$ and $f_2 = 1$ in the equality thus getting $\omega_{1y} = \omega_{2y}$. Integrating this PDE yields

$$\omega_1 = \omega_2 + h(t, x)$$

with an arbitrary smooth function h. In view of this equation, relation (13) takes the form

$$2A_{1\lambda_i} + A_{2\lambda_i} = 0 \qquad i = 1, 2.$$

Hence we conclude that there exists a function $\Lambda(\lambda_1, \lambda_2)$ such that

$$A_1 = \Lambda(\lambda_1, \lambda_2) + \tilde{A}_1(\omega_1) \qquad A_2 = -2\Lambda(\lambda_1, \lambda_2) + \tilde{A}_2(\omega_2). \tag{15}$$

Within the equivalence transformation (9) with properly chosen functions h_1, h_2 we can put $\tilde{A}_1(\omega_1) = 0$, $\tilde{A}_2(\omega_2) = 0$. Furthermore, defining the new separation constants as

$$\lambda_1' = \Lambda(\lambda_1, \lambda_2) \qquad \lambda_2' = \lambda_2$$

and omitting the primes we represent (15) in the form

$$A_1 = \lambda_1 \qquad A_2 = -2\lambda_1.$$

Consequently, system (6) takes the form

$$\begin{aligned} &\dot{\varphi}_0(t) = A_0(t)\varphi_0(t) \\ &\dot{\varphi}_1(\omega_1) = \lambda_1\varphi_1(\omega_1) \\ &\ddot{\varphi}_2(\omega_2) = -2\lambda_1\dot{\varphi}_2(\omega_2) + A_3(\omega_2, \lambda_1, \lambda_2)\varphi_2(\omega_2). \end{aligned} \tag{16}$$

Making the change of variables $\varphi_2 = \phi \exp\{-\lambda_1\omega_2\}$ reduces the third equation of system (16) to

$$\ddot{\phi} = (\lambda_1^2 + A_3)\phi.$$

Let $\phi = \phi(\omega_2, \lambda_1, \lambda_2)$ be a solution of this equation. Then, the corresponding solution with separated variables becomes

$$u = Q(t, x, y)\phi(\omega_2, \lambda_1, \lambda_2) \exp\left\{\int A_0(t)\,\mathrm{d}t + \lambda_1(\omega_1 - \omega_2)\right\}.$$

The structure of so obtained solution with separated variables is such that dependence of ω_1 on y is not essential. Indeed, the function ω_1 enters into the solution only as a combination $\omega_1 - \omega_2$ and the latter is equal to $h(t, x)$. Consequently, we have proved that without loss of generality we may choose $\omega_{1y} = 0$.

Given the condition $\omega_{1y} = 0$, equation (13) reduces to the relations $A_{2\lambda_i} = 0$, $i = 1, 2$, hence we get $A_2 = A_2(\omega_2)$. Choosing appropriately the function h_2 in (9) we can put $A_2 = 0$. Next, differentiating (12) with respect to λ_i we arrive at the equations

$$A_{0\lambda_i} + A_{1\lambda_i}(\omega_{1t} + y\omega_{1x}) = \nu A_{3\lambda_i}\omega_{2y}^2 \qquad i = 1, 2. \tag{17}$$

Differentiating twice these equations with respect to y yields

$$A_{3\omega_2\omega_2\lambda_i}\omega_{2y}^4 + 5A_{3\omega_2\lambda_i}\omega_{2y}^2\omega_{2yy} + 2A_{3\lambda_i}(\omega_{2yy}^2 + \omega_{2y}\omega_{2yyy}) = 0 \tag{18}$$

where $i = 1, 2$.

Note that due to (17) the inequality $A_{3\lambda_i} \neq 0$ holds. Dividing (18) into $A_{3\lambda_i}$ and differentiating the equality obtained by λ_j, $j = 1, 2$ we get

$$\left(\frac{A_{3\omega_2\omega_2\lambda_i}}{A_{3\lambda_i}}\right)_{\lambda_j}\omega_{2y}^2 + 5\left(\frac{A_{3\omega_2\lambda_i}}{A_{3\lambda_i}}\right)_{\lambda_j}\omega_{2yy} = 0 \qquad i, j = 1, 2. \tag{19}$$

Case 1. At least one of the four expressions

$$\left(\frac{A_{3\omega_2\lambda_i}}{A_{3\lambda_i}}\right)_{\lambda_j}$$

does not vanish. Then it is easy to become convinced that

$$\frac{\omega_{2yy}}{\omega_{2y}^2} = f(\omega_2)$$

holds true. Integration of this relation yields

$$\omega_{2y} = g_1(t, x) \exp\left\{\int f(\omega_2)\,\mathrm{d}\omega_2\right\}$$

where $g_1(t, x)$ is an arbitrary smooth function.

Next, by using the equivalence relation (8) we reduce the equation obtained to the form

$$\omega_{2y} = g_1(t, x)$$

whence

$$\omega_2 = y g_1(t, x) + g_2(t, x) \tag{20}$$

$g_2(t, x)$ being an arbitrary smooth function.

In view of this result, (18) takes the form $A_{3\omega_2\omega_2\lambda_i} = 0$, $i = 1, 2$, whence

$$A_3 = \Lambda_1(\lambda_1, \lambda_2)\omega_2 + \Lambda_2(\lambda_1, \lambda_2) + F(\omega_2) \tag{21}$$

where Λ_1, Λ_2, F are arbitrary smooth functions of the indicated variables. Furthermore, it is not difficult to prove that Λ_1, Λ_2 are functionally independent (since otherwise condition (5) would be broken) and, consequently, after redefining λ_1, λ_2 we can represent (21) in the form

$$A_3 = \lambda_1\omega_2 + \lambda_2 + F(\omega_2). \tag{22}$$

Case 2. Suppose that now

$$\left(\frac{A_{3\omega_2\lambda_i}}{A_{3\lambda_i}}\right)_{\lambda_j} = 0 \qquad i, j = 1, 2.$$

Integrating this system of PDEs gives the following form of $A_{3\lambda_i}$

$$A_{3\lambda_i} = B_i(\omega_2)L_i(\lambda_1, \lambda_2) \qquad i = 1, 2 \tag{23}$$

where B_i, L_i are arbitrary smooth functions and $B_1^2 + B_2^2 \neq 0$.

As a compatibility condition of system (23) we get

$$B_1 L_{1\lambda_2} = B_2 L_{2\lambda_1}.$$

Subcase 2.1. $L_{1\lambda_2} \neq 0$, $L_{2\lambda_1} \neq 0$. Given these restrictions the compatibility condition is transformed to

$$\frac{B_1(\omega_2)}{B_2(\omega_2)} = \frac{L_{2\lambda_1}}{L_{1\lambda_2}} = \text{ constant.} \tag{24}$$

Integrating system (23) with the result of (24) yields

$$A_3 = \Lambda(\lambda_1, \lambda_2)F_1(\omega_2) + F_2(\omega_2)$$

where Λ, F are arbitrary smooth functions of the indicated variables. After redefining the separation parameters λ_1, λ_2 we represent the relation as follows

$$A_3 = \lambda_1 F_1(\omega_2) + F_2(\omega_2). \tag{25}$$

Subcase 2.2. $L_{1\lambda_2} = 0$, $L_{2\lambda_1} = 0$. Integrating system (23) and redefining the separation parameters λ_1, λ_2 yields

$$A_3 = \lambda_1 S_1(\omega_2) + \lambda_2 S_2(\omega_2) + S_0(\omega_2) \tag{26}$$

where S_1, S_2, S_0 are arbitrary smooth functions. An analysis of formulae (22), (25) and (26) shows that the first two are particular cases of formula (26). Thus, the most general form of the function A_3 is given by (26).

Inserting (26) into (17) and differentiating the equality obtained with respect to x and λ_j gives $A_{1\lambda_i\lambda_j} = 0$, $i, j = 1, 2$. Hence, we get for A_1

$$A_1 = \lambda_1 L_1(\omega_1) + \lambda_2 L_2(\omega_1) + L_0(\omega_1) \tag{27}$$

where L_1, L_2, L_0 are arbitrary smooth functions.

Next, inserting (26), (27) into (17) and differentiating the equation obtained with respect to λ_j we get $A_{0\lambda_i\lambda_j} = 0$, $i, j = 1, 2$, hence

$$A_0 = \lambda_1 R_1(t) + \lambda_2 R_2(t) + R_0(t) \tag{28}$$

where R_1, R_2, R_0 are arbitrary smooth functions.

With these results we can split equations (11) and (12) by λ_1, λ_2 thus obtaining a system of four nonlinear PDEs for the three functions ω_1, ω_2, Q

$$Q\omega_{2t} + yQ\omega_{2x} = (\nu y + kx)Q\omega_{2y} + 2\nu Q_y\omega_{2y} + \nu Q\omega_{2yy} \tag{29}$$

$$Q_t + QR_0 + QL_0(\omega_{1t} + y\omega_{1x}) + yQ_x = \nu Q + (\nu y + kx)Q_y + \nu Q_{yy} + \nu QS_0\omega_{2y}^2 \tag{30}$$

$$R_1 + L_1(\omega_{1t} + y\omega_{1x}) = \nu S_1\omega_{2y}^2 \tag{31}$$

$$R_2 + L_2(\omega_{1t} + y\omega_{1x}) = \nu S_2\omega_{2y}^2. \tag{32}$$

Making an equivalence transformation (9) with appropriately chosen functions we can put $L_0 = 0$ and $R_0 = 0$. Next, due to the requirement in (5) $S_1S_2 \neq 0$.

There are two inequivalent cases $L_2 = 0$ and $L_2 \neq 0$. Since they are handled in a similar way, we consider in detail the case $L_2 = 0$ only. In view of (5) L_1 does not vanish. Choosing appropriately the functions f_1, f_2 in (8) we can put $L_1 = 1$, $S_2 = \pm 1$ in formulae (29)–(32). Integrating (32) with (31) yields

$$\omega_2 = R(t)y + F(t,x) \qquad R(t) \neq 0 \tag{33}$$

where R, F are arbitrary smooth functions and $R_2 = \pm\nu R^2$.

Differentiating (31) twice with respect to y and taking into account (33) we arrive at the equation $S_{1\omega_2\omega_2} = 0$, therefore

$$S_1 = C_1\omega_2 + C_2$$

where $C_1 \neq 0$ and C_2 are arbitrary constants. Next, integrating (31) we obtain for ω_1, $F(t,x)$

$$\omega_1 = \nu C_1(R^3x + P(t)) - \int R_1(t)\,\mathrm{d}t$$
$$F(t,x) = 3\dot{R} + R^{-2}\dot{P}(t) - C_1^{-1}C_2 \tag{34}$$

where $P(t)$ is an arbitrary smooth function.

Hence, we conclude that the corresponding solution with separated variables reads as

$$\begin{aligned} u &= Q(t,x,y)\exp\left\{\lambda_1\int R_1(t)\,\mathrm{d}t + \lambda_2\int R_2(t)\,\mathrm{d}t\right\}\exp\{\lambda_1\omega_1\}\varphi_2(\omega_2) \\ &= Q(t,x,y)\exp\left\{\lambda_2\int R_2(t)\,\mathrm{d}t\right\}\exp\{\lambda_1(\nu C_1(R^3x + P(t)))\}\varphi_2(\omega_2). \end{aligned}$$

Thus, $R_1(t)$ does not enter into the solution with separated variables and, therefore, we can put $R_1 = 0$ in (34). Furthermore, within an equivalence transformation (8) we can choose $C_1 = \nu^{-1}$, $C_2 = 0$, thus getting

$$\omega_1 = R(t)^3x + P(t) \tag{35}$$

$$\omega_2 = R(t)y + 3\dot{R}(t)x + \dot{P}R(t)^{-2}. \tag{36}$$

Provided $L_2 \neq 0$, the forms of the functions ω_1, ω_2 are the same as those given in (35), (36).

Inserting (35) and (36) into (29) and integrating by y we get the form of the factor $Q(t,x,y)$

$$Q = \exp\left\{\left(\frac{4\dot{R} - \nu R}{4\nu R}\right)y^2 + \left(\frac{3\ddot{R} - kR}{2\nu R}\right)xy + \frac{y}{2\nu R}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{P}}{R^2}\right) + M(t,x)\right\}. \tag{37}$$

Substituting (37) into (30) we come to the following relation

$$\frac{1}{\nu}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot R}{R}\right)y^2+\frac{3}{2\nu}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\ddot R}{R}\right)xy+\frac{1}{2\nu}\dot Z y+M_t+\frac{1}{2\nu}\left(3\frac{\ddot R}{R}-k\right)y^2+yM_x$$
$$=\frac{\nu}{2}+2\frac{\dot R}{R}+(\nu y+kx)\left(\left(\frac{2\dot R}{\nu R}-\frac{1}{2}\right)y+\frac{1}{2\nu}\left(3\frac{\ddot R}{R}-k\right)x+\frac{1}{2\nu}Z\right)$$
$$+\nu\left(\left(\frac{2\dot R}{\nu R}-\frac{1}{2}\right)y+\frac{1}{2\nu}\left(3\frac{\ddot R}{R}-k\right)x+\frac{1}{2\nu}Z\right)^2+\nu S_0R^2 \qquad (38)$$

where we use the notation

$$Z(t)=R^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot P}{R^2}\right).$$

Differentiating (38) three times with respect to y yields $S_{0\omega_2\omega_2\omega_2}=0$, therefore

$$S_0=C_1\omega_2^2+C_2\omega_2+C_3$$

where C_1, C_2, C_3 are arbitrary constants. Next, differentiating (38) with respect to y twice and with respect to x once we get $M_{xxx}=0$ or

$$M=M_1(t)x^2+M_2(t)x+M_3(t)$$

where M_1, M_2, M_3 are arbitrary smooth functions.

Finally, inserting the obtained expressions for S_0, M into (38) and splitting the variables x, y we come to the following system of ODEs

$$\frac{\ddot R}{R}=2\frac{\dot R^2}{R^2}+\frac{2\nu^2}{5}C_1R^4-\frac{\nu^2}{10}+\frac{k}{5} \qquad (39)$$

$$M_1=-\frac{3}{4\nu}\frac{\dddot R}{R}+\frac{15}{4\nu}+3\nu C_1\dot R R^3-\frac{k}{4} \qquad (40)$$

$$\dot M_1=\frac{9\ddot R^2}{4\nu R^2}+9\nu C_1R^2\dot R^2-\frac{k^2}{4\nu} \qquad (41)$$

$$M_2=-\frac{1}{2\nu}\dot Z+\frac{2\dot R}{\nu R}Z+\nu C_2R^3+2\nu R\dot P C_1 \qquad (42)$$

$$\dot M_2=\frac{3\ddot R}{2\nu R}Z+3\nu R^2\dot R C_2+6\nu\dot R\dot P C_1 \qquad (43)$$

$$\dot M_3=\frac{\nu}{2}+2\frac{\dot R}{R}+\frac{1}{4\nu}Z^2+\nu C_1\frac{\dot P^2}{\dot R^2}+\nu C_2\dot P+\nu C_3R^2. \qquad (44)$$

Differentiating (40) with respect to t and subtracting the resulting equation from (41) yields the fourth-order ODE for the function R

$$-\frac{R^{(IV)}}{R}+6\frac{\dot R\dddot R}{R^3}+2\frac{\ddot R^2}{R^2}-10\frac{\dot R^2\ddot R}{R^3}+4\nu^2C_1\ddot R R^3+\frac{k^3}{3}=0.$$

Reducing the order of this ODE with the help of equation (39) and its first- and second-order differential consequences we arrive at the following relation

$$\frac{4\nu^2}{25}C_1^2R^8=\frac{\nu^4}{100}+\frac{k^2}{25}-\frac{\nu^2k}{25}-\frac{k^2}{9}. \qquad (45)$$

If in (45) $C_1\neq 0$, then in view of (39) $k=0$. Provided, $C_1=0$, k is a root of the quadratic equation

$$64k^2+36\nu^2k-9\nu^2=0$$

hence $k=3\nu^2/16$ or $k=-3\nu^2/4$.

Thus the system of ODEs (39)–(44) is consistent only if the parameter k takes one of three values 0, $3\nu^2/16$, $-3\nu^2/4$. Consequently, KE (2) has solutions with separated variables in the case considered (i.e. provided the system (4) takes the form (6)) only for the values of the parameter k given previously. This provides the proof of the first part of theorem 1.

We examine the three possible cases 0, $3\nu^2/16$, $-3\nu^2/4$ separately.

Case 1. For $k = 0$, the equality $R(t) = \pm 2^{-1/2} S_1^{-1/4}$ = constant holds. We denote this constant as r. Next, it follows from (43) that $M_2 = m =$ constant. In view of these facts we get from (42) the ODE for $P(t)$

$$-\dddot{P} + \nu^2 \dot{P} + 2\nu r^3(\nu S_2 r^3 - m) = 0$$

which general solutions reads

$$P(t) = C_4\,\mathrm{e}^{\nu t} + C_5\,\mathrm{e}^{-\nu t} + 2r^3(m\nu^{-1} - S_2 r^3)t + C_6 \tag{46}$$

where C_4, C_5, C_6 are arbitrary constants.

A direct check shows that by applying finite transformations from the symmetry group admitted by KE under $k = 0$ to the obtained solution with separated variables (3), (35), (36) and (46) we can cancel $P(t)$.

Scaling when necessary ω_1, ω_2 in (35), (36) we can choose $r = 1$. Hence we get the equality $C_1 = 1/4$. Summing up we conclude that the following relations hold

$$Q = \exp\left(-\frac{y^2}{4} + \nu C_2 x + \nu\left(C_3 + \frac{1}{2}\right)t\right) \qquad \omega_1 = x \qquad \omega_2 = y$$

$$\dot{\varphi}_0 = \nu\left(\lambda_1 - \left(C_3 + \frac{1}{2}\right)\right)\varphi_0$$

$$\dot{\varphi}_1 = \nu(\lambda_2 - C_2)\varphi_1 \qquad \ddot{\varphi}_2 = \left(\frac{\omega_2^2}{4} + \lambda_2\omega_2 + \lambda_1 - \frac{1}{2}\right)\varphi_2.$$

Then, the corresponding solution with separated variables is

$$u = \varphi_2 \exp\left(-\frac{y^2}{4} + \nu(\lambda_1 t + \lambda_2 x)\right).$$

Consequently, the constants C_2 and $C_3 + (1/2)$ do not enter the final form of the solution with separated variables. This means that we can put $C_2 = 0$ and $C_3 = -(1/2)$.

Thus we have proved the validity of the first part of theorem 5.

Cases 2 and 3. For $k = (3\nu^2/16)$ or $k = -(3\nu^2/4)$. In these cases we get from (39)

$$\frac{\ddot{R}}{R} - 2\left(\frac{\dot{R}}{R}\right)^2 = -a^2$$

where

$$a = \begin{cases} \dfrac{\nu}{4} & \text{under } k = \dfrac{3\nu^2}{16} \\ \dfrac{\nu}{2} & \text{under } k = -\dfrac{3\nu^2}{4}. \end{cases}$$

Integrating the ODEs yields

$$R(t) = (C_1 \sinh at + C_2 \cosh at)^{-1}$$

where C_1, C_2 are arbitrary constants.

Using shifts with respect to t and the equivalence transformation (8) we get the four inequivalent forms of the function $R(t)$

$$R(t) = \frac{1}{\cosh at} \qquad R(t) = \frac{1}{\sinh at} \qquad R(t) = \exp\{\pm at\}.$$

Comparing (42) and the first-order differential consequence of (43) yields the second-order ODE for $Z(t) = R^{-1}(\mathrm{d}/\mathrm{d}t)/(\dot{P}/R^2)$

$$-\ddot{Z} + 4\frac{\dot{R}}{R}\dot{Z} + \left(\frac{\ddot{R}}{R} - 4\frac{\dot{R}}{R}\right) Z = 0. \tag{47}$$

The general solution of this equation has the following structure

$$Z(t) = C_1 Z_1(t) + C_2 Z_2(t)$$

where C_1, C_2 are integration constants. Hence, we conclude that the function $P(t)$ is of the form

$$P(t) = C_1 P_1(t) + C_2 P_2(t) + C_3 P_3(t) + C_4 P_4(t) \tag{48}$$

where C_3, C_4 are integration constants.

On the other hand, if we apply to the solution with separated variables (3), (35), (36) with $P(t) = 0$ finite transformations from the symmetry group of KE under $k = (3\nu^2/16)$ or $k = -(3\nu^2/4)$, then we get an equivalent solution with separated variables such that $P(t)$ is of the form

$$P(t) = C_1' P_1'(t) + C_2' P_2'(t) + C_3' P_3'(t) + C_4' P_4'(t). \tag{49}$$

Here $C_1', \ldots, C_4'$ are arbitrary constants and the functions $P_1'(t), \ldots, P_4'(t)$ are linearly independent. Hence we conclude that due to the theorem on the existence and uniqueness of the Cauchy problem for a fourth-order ODE (47) (considered as an equation for the function $P(t)$) the expressions on the right-hand sides of (48) and (49) coincide within the choice of constants $C_i, C_i', i = 1, \ldots, 4$. Consequently, without loss of generality we can put $P(t) = 0$ in formulae (35) and (36).

Using the reasonings analogous to those of case 1 we can put $r = 1$, $C_2 = 0$, $C_3 = 0$. The second part of theorem 6 is thus proved.

A similar analysis of the separability of KE into three ODEs (7) yields the proofs of the remaining assertions from section 2.

4. Exact solutions

Remarkably, for the equation under study it is possible to give a complete account of solutions with separated variables. For the case when KE separates into three first-order ODEs (7), we get the following family of its exact solutions

$$\begin{aligned} u = \exp\Bigg\{ & \nu \int \left(\frac{f_1\lambda_1 + f_2\lambda_2}{\dot{f}_2 f_1 - \dot{f}_1 f_2}\right)^2 \mathrm{d}t + \lambda_1 \frac{f_1 y - \dot{f}_1 x}{\dot{f}_2 f_1 - \dot{f}_1 f_2} + \lambda_2 \frac{f_2 y - \dot{f}_2 x}{\dot{f}_2 f_1 - \dot{f}_1 f_2} \\ & + \left(-\frac{1}{4\nu}\frac{\ddot{f}_2 f_1 - \ddot{f}_1 f_2}{\dot{f}_2 f_1 - \dot{f}_1 f_2} - \frac{1}{4}\right) y^2 + \frac{1}{2\nu}\left(\frac{\ddot{f}_2 \dot{f}_1 - \ddot{f}_1 \dot{f}_2}{\dot{f}_2 f_1 - \dot{f}_1 f_2} - k\right) xy \\ & + \left(\frac{1}{4\nu}\frac{\dddot{f}_2\, \dot{f}_1 - \dddot{f}_1\, \dot{f}_2}{\dot{f}_2 f_1 - \dot{f}_1 f_2} - \frac{k}{4}\right) x^2 - \frac{1}{2}\ln|\dot{f}_2 f_1 - \dot{f}_1 f_2| + \frac{\nu}{2} t \Bigg\} \end{aligned}$$

where k, $f_1(t)$, $f_2(t)$ are given by the corresponding formulae from theorems 2–5.

Next, for the case when KE separates into three ODEs of the form (6) we obtain the following families of its exact solutions:

(1) $k = 0$ (this case has been considered in theorem 5)

$$u = \exp\left(-\frac{y^2}{4} + \nu(\lambda_1 t + \lambda_2 x)\right) D_{\lambda_2^2-\lambda_1}(y + 2\lambda_2)$$

where D_ν is the parabolic cylinder function.

(2) $k = 3\nu^2/16$ or $k = -3\nu^2/4$

$$u = \exp\left\{\nu\lambda_1 \int R^2\,\mathrm{d}t + \nu\lambda_2 R^3 x\left(\frac{\dot{R}}{\nu R} - \frac{1}{4}\right)y^2 + \frac{1}{2\nu}\left(3\frac{\ddot{R}}{R} - k\right)xy\right.$$
$$\left. + \left(-\frac{3}{4\nu R}\dddot{R} + \frac{15\dot{R}\ddot{R}}{4\nu R^2} - \frac{k}{4}\right)x^2 + \frac{\nu}{2}t + 2\ln R\right\}\{\lambda_2(Ry + 3\dot{R}x)$$
$$+\lambda_1\}^{1/2} Z_{1/3}\left(\frac{2}{3\lambda_2}(\lambda_2(Ry + 3\dot{R}x) + \lambda_1)^{3/2}\right)$$

where R is given by the corresponding formula from theorem 6 and $Z_{1/3}$ is the cylindrical function.

Note that the obtained families of exact solutions of KE contain two continuous parameters λ_1, λ_2. These parameters have the meaning of eigenvalues of two commuting symmetry operators of KE, while the corresponding solution with separated variables is the eigenfunction of these operators. Provided some appropriate boundary and initial conditions are imposed, the parameters become discrete and thus we get a basis for expanding sufficiently smooth solutions of KE into series.

5. Conclusions

It is a remarkable feature of the KE (2) that a classical problem of variable separation can be solved in full generality. The results obtained in this way are in good correspondence with the ones on symmetry classification of KEs of the form (2). As follows from [5, 6], the cases $k = 3\nu^2/16$ and $k = -3\nu^2/4$ are distinguished by the fact that the corresponding KEs (2) admit the most extensive symmetry groups. For these choices of k, KE (2) is invariant with respect to eight-parameter Lie transformation groups, while for all other values of k the maximal group is six-parameter.

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